BV Action of the R-Poisson Sigma Model

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- Topological field theories
  - A/B models [Witten '88]
  - Chern-Simons theory
  - Dirac Sigma model [Kotov, Schaller, Strobl '04]
  - Quantum matter
- BV qunatization for topological field theories by AKSZ construction [AKSZ '95]
  - Requires the QP-structure on the target
  - Presence of the Wess-Zumino term breaks the QP-structure
- How to construct BV action for topological field theories that do not allow QP-structure?

Twisted Poisson Sigma model is a 2-dimensional topological sigma model with the action:

[Ikeda '93] [Schaller, Strobl '94] [Klimcik, Strobl '02]

$$S = \int_{\Sigma_2} \left(A_i \wedge \mathrm{d}X^i + \frac{1}{2}\Pi^{ij}(X) A_i \wedge A_j \right) + \int_{\Sigma_3} X^* H_3 \,,$$

where $X : \Sigma_2 \to M$ and $A \in \Omega^1(\Sigma_2, X^* T^* M)$.
The target space M forms a twisted Poisson manifold:

$$[\Pi, \Pi] = 2\langle \otimes^3 \Pi, H_3 \rangle \,, \qquad \mathrm{d}H_3 = 0 \,.$$

Gauge transformations:

$$\begin{aligned} \delta X^i &= -\Pi^{ij}\epsilon_j \,, \\ \delta A_i &= \mathrm{d}\epsilon_i - \partial_i \Pi^{jk}\epsilon_j A_k + \frac{1}{2}\Pi^{jl} H_{ikl}\epsilon_j (\mathrm{d}X^k - \Pi^{km} A_m) \,. \end{aligned}$$

Equations of motion:

$$\begin{aligned} F^i &= \mathrm{d}X^i - \Pi^{ij} A_j = 0 \,, \\ G_i &= \mathrm{d}A_i + \frac{1}{2}\partial_i \Pi^{jk} A_j \wedge A_k + \frac{1}{2} H_{ijk}\, \mathrm{d}X^j \wedge \mathrm{d}X^k = 0 \end{aligned}$$

Topological sigma model in $p+1$ dimensions formed as a generalization of the Poisson sigma model:

[Chatzistavrakidis '21]

$$S = \int_{\Sigma_{p+1}} \Big(Z_i \wedge \mathrm{d}X^i - A_i \wedge \mathrm{d}Y^i + \Pi^{ij}(X) Z_i \wedge A_j - \frac{1}{2}\partial_k \Pi^{ij}(X) Y^k \wedge A_i \wedge A_j + \\ + \frac{1}{(p+1)!} R^{i_1 \dots i_{p+1}}(X) A_{i_1} \wedge \ldots \wedge A_{i_{p+1}} \Big) + \int_{\Sigma_{p+2}} X^* H_{p+2} ,$$

where $A \in \Omega^1(\Sigma_{p+1}, X^*T^*M)$, $Y \in \Omega^{p-1}(\Sigma_{p+1}, X^*T^*M)$, $Z \in \Omega^p(\Sigma_{p+1}, X^*T^*M)$. Target space M forms a twisted R-Poisson structure:

$$[\Pi, \Pi] = 0 , \qquad [\Pi, R] = (-1)^{p+1} \langle \otimes^{p+2} \Pi, H \rangle , \qquad \mathrm{d}H_{p+2} = 0$$

Equations of motion:

$$\begin{aligned} F^i &= \mathrm{d}X^i + \Pi^{ij} A_j , \\ G_i &= \mathrm{d}A_i + \frac{1}{2}\partial_i \Pi^{jk} A_j \wedge A_k , \\ \mathcal{F}^i &= \mathrm{d}Y^i + \ldots , \\ \mathcal{G}_i &= (-1)^{p+1} \mathrm{d}Z_i + \ldots \end{aligned}$$

Gauge transformations with 3 gauge parameters ϵ, χ, ψ:

$$\begin{aligned}
\delta X^i &= -\Pi^{ij}\epsilon_j \, , \\
\delta A_i &= \mathrm{d}\epsilon_i - \partial_i \Pi^{jk}\epsilon_j A_k \, , \\
\delta Y^i &= (-1)^{p-1}\,\mathrm{d}\chi^i - \Pi^{ij}\psi_j - \partial_k \Pi^{ij}(\chi^k A_j + Y^k \epsilon_j) + \frac{1}{(p-1)!} R^{ijk_1\dots k_{p-1}}\epsilon_j A_{k_1}\dots A_{k_{p-1}} \, , \\
\delta Z_i &= (-1)^p\,\mathrm{d}\psi_i + \partial_i \Pi^{jk}\left(Z_j\epsilon_k + \psi_j A_k\right) - \partial_i\partial_j \Pi^{kl}\left(Y^j A_k \epsilon_l - \frac{1}{2} A_k A_l \chi^j\right) + \\
&\quad + \frac{(-1)^p}{p!}\, f_i^{jk_1\dots k_p}\epsilon_j A_{k_1}\dots A_{k_p} + \\
&\quad + \frac{1}{(p+1)!}\Pi^{jm} H_{imk_1\dots k_p}\epsilon_j \sum_{r=1}^{p}(-1)^r \binom{p+1}{r} \prod_{s=1}^{r} F^{k_s} \prod_{t=r+1}^{p} \Pi^{k_t l_t} A_{l_t} \, ,
\end{aligned}$$

where:

$$f_i^{j_1\dots j_{p+1}} = \partial_i R^{j_1\dots j_{p+1}} + H_i^{j_1\dots j_{p+1}} = \partial_i R^{j_1\dots j_{p+1}} + \Pi^{j_1 k_1}\dots \Pi^{j_{p+1}k_{p+1}} H_{ik_1\dots k_{p+1}}$$

QP-manifolds

Q-manifold: graded differential manifold with a cohomological vector field Q of degree 1

P-manifold: graded differential manifold equipped with a graded symplectic form ω

Twisted R-Poisson manifolds are both Q-manifolds and P-manifolds:

$$
\begin{aligned}
Q &= \Pi^{ji} a_j \partial_i - \frac{1}{2}\partial_{x^i}\Pi^{jk} a_j a_k \partial_{a_i} + \left((-1)^p \Pi^{ji} z_j - \partial_j \Pi^{ik} a_k y^j + \frac{1}{p!} R^{ij_1\dots j_p} a_{j_1}\dots a_{j_p}\right)\partial_{y^i} + \\
&\quad + \left(\partial_i \Pi^{jk} a_k z_j - \frac{(-1)^p}{2}\partial_i\partial_j \Pi^{kl} y^j a_k a_l + \frac{(-1)^p}{(p+1)!} f_i^{k_1\dots k_{p+1}} a_{k_1}\dots a_{k_{p+1}}\right)\partial_{z_i} \\
\omega &= \mathrm{d}x^i \wedge \mathrm{d}z_i + \mathrm{d}a_i \wedge \mathrm{d}y^i
\end{aligned}
$$

QP-manifold: Q-manifold equipped with a symplectic form ω that satisfies the compatibility condition with the Q vector:

$$\mathcal{L}_Q \omega = 0\,.$$

But for twisted R-Poisson manifold this is satisfied only if $H = 0$.

BV construction

General procedure:

- Enlarge the field content of the theory by ghost fields (and ghosts for ghosts).
- Introduce antifields for all the fields and ghosts such that the sum of ghost degrees of the field and its antifield is -1.
- Define a symplectic structure on the space of fields and antifields that defines the antibracket $(\cdot, \cdot)$.
- Construct the BV action S_{BV} such that it satisfies the classical master equation $(S_{BV}, S_{BV}) = 0$ and it reduces to the classical action S_0 when all the antifields are set to 0.

Field/ghost/antifield content for 3-dimensional twisted R-Poisson sigma model:

Field	X^i	A_i	Y^i	Z_i	ϵ_i	χ^i	ψ_i	$\widetilde{\psi}_i$
Form degree	0	1	1	2	0	0	1	0
Ghost degree	0	0	0	0	1	1	1	2

Field	X^i_+	A^+_i	Y^i_+	Z^+_i	ϵ^+_i	χ^i_+	ψ^+_i	$\widetilde{\psi}^i_+$
Form degree	3	2	2	1	3	3	2	3
Ghost degree	-1	-1	-1	-1	-2	-2	-2	-3

There are several methods that can be used to construct the BV action:

- AKSZ procedure, but it works only for the QP-manifolds.
- Expand the BV action by the number of antifields:

$$S_{BV} = S_0 + S_1 + S_2 + \dots ,$$

  and solve the classical master equation order by order. [Ikeda, Strobl '19]
- Construct the BV operator $s = (S_{BV}, \cdot)$ instead of the BV action directly.

The BV operator needs to satisfiy 2 properties:

- It is nilpotent ($s^2 = 0$).
- When antifields are set to 0, it reduces to the BRST operator s_0.

BRST operator

The action of the BRST operator on the fields from classical theory follows from gauge transformations:

$$\begin{aligned}
s_0 X^i &= -\Pi^{ij}\epsilon_j\,, \\
s_0 A_i &= \mathrm{d}\epsilon_i - \partial_i\Pi^{jk}\epsilon_j A_k\,, \\
s_0 Y^i &= -\mathrm{d}\chi^i - \Pi^{ij}\psi_j - \partial_k\Pi^{ij}(A_j\chi^k + \epsilon_j Y^k) + R^{ijk}\epsilon_j A_k\,, \\
s_0 Z_i &= \mathrm{d}\psi_i + \partial_i\Pi^{jk}(A_j\psi_k - \epsilon_j Z_k) - \partial_i\partial_l\Pi^{jk}\left(\epsilon_j A_k Y^l - \frac{1}{2}A_j A_k\chi^l\right) + \\
&\quad + \frac{1}{2}f_i^{jkl}\epsilon_j A_k A_l + \frac{1}{2}H_{il}{}^{jk}\epsilon_j A_k F^l + \frac{1}{6}H_{ikl}{}^{j}\epsilon_j F^k F^l
\end{aligned}$$

Its action on the ghosts follows partly from the AKSZ construction and partly from the requirment that the BRST has to be nilpotent on-shell:

$$\begin{aligned}
s_0\epsilon_i &= -\frac{1}{2}\partial_i\Pi^{jk}\epsilon_j\epsilon_k\,, \\
s_0\chi^i &= -\partial_k\Pi^{ij}\epsilon_j\chi^k - \Pi^{ij}\widetilde{\psi}_j - \frac{1}{2}R^{ijk}\epsilon_j\epsilon_k\,, \\
s_0\widetilde{\psi}_i &= -\partial_i\Pi^{jk}\epsilon_j\widetilde{\psi}_k - \frac{1}{2}\partial_i\partial_l\Pi^{jk}\epsilon_j\epsilon_k\widetilde{s}\chi^l - \frac{1}{6}f_i^{jkl}\epsilon_j\epsilon_k\epsilon_l\,, \\
s_0\psi_i &= \mathrm{d}\widetilde{\psi}_i + \partial_i\Pi^{jk}(-\epsilon_j\psi_k + A_j\widetilde{\psi}_k) - \partial_i\partial_l\Pi^{jk}\left(\epsilon_j A_k\chi^l + \frac{1}{2}\epsilon_j\epsilon_k Y^l\right) + \frac{1}{2}f_i^{jkl}\epsilon_j\epsilon_k A_l\,.
\end{aligned}$$

When the s_0^2 acts on the fields, it gives terms that contain EOM (so that it vanishes on-shell).

Then we construct operator s_1 that is equal to s_0 + the terms that contain antifields which would cancel EOM terms in s_0^2. This can easily be done because:

$$
\begin{aligned}
&sZ_+^i \supset -F^i\,, && sY_i^+ \supset G_i\,, && sA_+^i \supset \mathcal{F}^i\,, && sX_i^+ \supset \pm\mathcal{G}_i\,, \\
&s\chi_i^+ \supset \mathrm{d}Y_i^+\,, && s\psi_+^i \supset \mathrm{d}Z_+^i\,, && s\widetilde{\psi}_+^i \supset \mathrm{d}\psi_+^i\,, && s\epsilon_+^i \supset \pm X_i^+\,.
\end{aligned}
$$

For example:

$$
\begin{aligned}
s_0^2 A_i &= -\frac{1}{2}\partial_i\partial_l\Pi^{jk}\epsilon_j\epsilon_k F^l\,, \\
s_1 A_i &= s_0 A_i - \frac{1}{2}\partial_i\partial_l\Pi^{jk}\epsilon_j\epsilon_k Z_+^l\,.
\end{aligned}
$$

This refinement is repeated until a nilpotent operator $\widetilde{s}$ is constructed.

Nontrivial refinements of the fields:

$$\begin{aligned}
\widetilde{s}A_i &= s_0 A_i - \frac{1}{2}\partial_i\partial_l\Pi^{jk}\epsilon_j\epsilon_k Z_+^l\,, \\
\widetilde{s}Y^i &= s_0 Y^i + \partial_k\Pi^{ij}\widetilde{\psi}_j Z_+^k + \partial_k\partial_l\Pi^{ij}\epsilon_j\chi^k Z_+^l + \frac{1}{2}\partial_l R^{ijk}\epsilon_j\epsilon_k Z_+^l\,, \\
\widetilde{s}\psi_i &= s_0\psi_i + \partial_i\partial_l\Pi^{jk}\epsilon_j\widetilde{\psi}_k Z_+^l - \frac{1}{2}\partial_i\partial_l\partial_m\Pi^{jk}\epsilon_j\epsilon_k\chi^l Z_+^m + \frac{1}{6}\partial_m f_i^{jkl}\epsilon_j\epsilon_k\epsilon_l Z_+^m \\
\widetilde{s}Z_i &= s_0 Z_i - \partial_i\Pi^{jk} Y_j^+\widetilde{\psi}_k - \partial_i\partial_l\Pi^{jk}\left(\frac{1}{2}\epsilon_j\epsilon_k A_+^l - \epsilon_j\psi_k Z_+^l + A_j\widetilde{\psi}_k Z_+^l + \epsilon_j Y_k^+\chi^l - \epsilon_j\widetilde{\psi}_k\psi_+^l\right) \\
&+\partial_i\partial_l\partial_m\Pi^{jk}\left(\frac{1}{2}\epsilon_j\epsilon_k Y^l Z_+^m + \epsilon_j A_k\chi^l Z_+^m - \frac{1}{2}\epsilon_j\widetilde{\psi}_k Z_+^l Z_+^m + \frac{1}{2}\epsilon_j\epsilon_k\chi^l\psi_+^m\right) - \\
&-\frac{1}{4}\partial_i\partial_l\partial_m\partial_n\Pi^{jk}\epsilon_j\epsilon_k\chi^l Z_+^m Z_+^n - \frac{1}{2}f_i^{jkl}\epsilon_j\epsilon_k Y_l^+ - \frac{1}{2}\partial_m f_i^{jkl}\epsilon_j\epsilon_k A_l Z_+^m + \\
&+\frac{1}{6}\partial_m f_i^{jkl}\epsilon_j\epsilon_k\epsilon_l\psi_+^m - \frac{1}{6}\partial_m\partial_n f_i^{jkl}\epsilon_j\epsilon_k\epsilon_l Z_+^m Z_+^n + \ldots
\end{aligned}$$

This operator is now nilpotent/cohomological and in the zero antifield limit becomes the BRST operator.

But does there exist the BV action S_{BV} such that $\widetilde{s} = (S_{BV}, \cdot)$?

Another refinement

All extra refinements include equations of motion.

Some refinements are not allowed beacuse:

- Form and ghost degrees wouldn't match properly.
- They would spoil 0-antifield limit.
- There is no way for the extra terms to produce a nilpotent operator.

In addition, some refinements are dependent at the end only one independent possibility remain: refinement of $\widetilde{s}\psi$ by a $\epsilon\epsilon F$ term.

With this new terms, the previously described refinement has to be carried out again, finally producing the desired BV operator s.

Nontrivial refinements:

$$
\begin{aligned}
s\psi^i &= \ldots + \frac{1}{4} H_{il}{}^{jk} \epsilon_j \epsilon_k F^l + \frac{1}{6} \partial_{(m} f_{i)}^{jkl} \epsilon_j \epsilon_k \epsilon_l Z_+^m \\
sY^i &= \ldots + \frac{1}{2} \left(\partial_l R^{ijk} + \frac{1}{2} H_l{}^{ijk} \right) \epsilon_j \epsilon_k Z_+^l \\
sZ_i &= \ldots + \partial_{(i} f_{m)}^{jkl} \left(\frac{1}{6} \epsilon_j \epsilon_k \epsilon_l \psi_+^m - \frac{1}{2} \epsilon_j \epsilon_k A_l Z_+^m \right) - \frac{1}{2} \left(\partial_i R^{jkl} + \frac{1}{2} H_i{}^{jkl} \right) \epsilon_j \epsilon_k Y_l^+ - \\
&\quad - \frac{1}{6} \partial_{(i} H_{m)l}{}^{jk} \epsilon_j \epsilon_k F^l Z_+^m - \left(\frac{1}{12} \partial_{(m} \partial_n f_{i)}^{jkl} + \frac{1}{8} \partial_{(m} \partial_n \Pi^{jp} H_{i)p}{}^{kl} \right) \epsilon_j \epsilon_k \epsilon_l Z_+^m Z_+^n
\end{aligned}
$$

Conclusion

- WZ terms break QP-structure and the AKSZ procedure cannot be used.
- The construction of the BV action has to be done in a different way and it is possible to do so by constructing the BV operator.

Outlook:

- Formulating a systematic procedure of constructing BV actions for topological field theories without the QP-structure.
- Construction of the quantum BV action.

Thank you for your attention!